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q-deformed Hermite polynomials in q-quantum mechanics

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Abstract. The q-special functions appear naturally in q-deformed quantum mechanics and both sides profit from this fact. Here we study the relation between the q-deformed harmonic oscillator and the q-Hermite polynomials. We discuss: recursion formula, generating function, Christoffel-Darboux identity, orthogonality relations and the moment functional.

1 Introduction

In the mathematical literature q-special functions have been studied very intensively [3]. It turns out that these functions appear in q-quantum mechanics when we try to diagonalize selfadjoint operators [2]. Due to the algebraic nature of q-quantum mechanics many properties of these systems can be derived from the algebraic structure. This in turn implies special properties of the respective q-special functions and would be hard to prove differently. A trivial example is the harmonic oscillator ($q = 1$) with its creation and annihilation operators on the algebraic side and the Hermite polynomials on the other.

In this short note we generalize this idea to the q-deformed harmonic oscillator where q-deformed Hermite polynomials appear in the eigenfunctions of the Hamiltonian [1]. A recursion formula for the q-Hermite polynomials follows directly from the construction of the eigenstates.

In Chap. 2 we solve this recursion formula explicitly and present a generating function of the q-Hermite polynomials. We also show that the Christoffel-Darboux identity follows from the recursion formula. We were, however, not able to prove the completeness of the polynomials with the help of this identity. From the work on the harmonic oscillator [1] we actually suspect that the q-Hermite polynomials are not a complete set of functions.

In Chap. 3 we give an explicit representation of the eigenstates of the q-harmonic oscillator in terms of the eigenstates of the coordinates. This then yields orthogonality relations for the q-Hermite polynomials which we derive in Chap. 4. It is interesting that there are two different measures by which the q-Hermite polynomials form an orthogonal set of functions. This again indicates the fact that the polynomials are not complete.

In Chap. 5 we use the matrix elements of powers of the coordinates to define a moment functional. As expected there are two different measures for this functional but the moment functional is independent of the choice of the measure.

Finally we use this moment functional to define an integral and we give its values in terms of the q-gamma function.

2 q-deformed Hermite polynomials

In the analysis of the q-deformed harmonic oscillator, as it was done in [1], the following recursion formula for the Hermite-polynomials occurs:

$$H_{n+1}^{(q)}(\xi) - q^{-\frac{1}{2}} q^{-2n} 2\xi H_n^{(q)}(\xi) + 2q^{-2}[n] H_{n-1}^{(q)}(\xi) = 0 \quad (2.1)$$

The q-number $[n]$ is defined as follows:

$$[n] \equiv [n]_{q^{-2}} = \frac{1-q^{-2n}}{1-q^{-2}} \quad (2.2)$$

We define the first two polynomials consistent with $H_{-1} = 0$:

$$H_0^{(q)}(\xi) = 1, \qquad H_1^{(q)}(\xi) = 2q^{-\frac{1}{2}}\xi \quad (2.3)$$

and obtain the next poynomials:

$$\begin{aligned} H_2^{(q)}(\xi) &= 4q^{-3}\xi^2 - 2q^{-2} \\ H_3^{(q)}(\xi) &= 8q^{-\frac{1}{2}}q^{-7}\xi^3 - 4q^{-\frac{1}{2}}q^{-2}[3]\xi \\ H_4^{(q)}(\xi) &= 16q^{-14}\xi^4 - 8q^{-5}[3](q^{-4}+1)\xi^2 + 4q^{-4}[3] \end{aligned} \quad (2.4)$$

A general expression is:

$$H_n^{(q)}(\xi) = q^{-\frac{n}{2}} \sum_{k=0}^{<\frac{n}{2}]} q^{-k} \frac{q^{-2\binom{n-2k}{2}} 2^{n-k}(-1)^k [n]!}{([2])^k [n-2k]! [k]_{q^{-4}}!} \xi^{n-2k} \quad (2.5)$$

The symbol $< \frac{n}{2}]$ means the largest integer smaller or equal $\frac{n}{2}$. In the limit $q \to 1$ we obtain from (2.5) the undeformed Hermite polynomials. A generating function for these q-Hermite polynomials has been found in [4]:

$$E_{q^{-2}}(\xi t) e_{q^{-4}}\left(t^2 \frac{q}{2(1-q^2)}\right) = \sum_{n=0}^{\infty} \frac{q^{\frac{n}{2}} 2^{-n} H_n^{(q)}(\xi)}{(q^{-2};q^{-2})_n} t^n \quad (2.6)$$

The exponential functions are defined as follows:

$$E_{q^{-2}}(t) = \sum_{n=0}^{\infty} \frac{q^{-2\binom{n}{2}}}{(q^{-2};q^{-2})_n} t^n$$
$$e_{q^{-4}}(t) = \sum_{n=0}^{\infty} \frac{t^n}{(q^{-4};q^{-4})_n} \tag{2.7}$$

In the classical theory of orthogonal polynomials the Christoffel-Darboux identity [5] is derived from a recursion relation.

The deduction of this identity for the q-deformed Hermite polynomials follows exactly the same steps as in the undeformed case. The result is:

$$\sum_{m=0}^{n} \frac{H_m(\xi_\mu^\sigma)H_m(\xi_\nu^\tau)}{2^m[m]!} = \frac{q^{\frac{1}{2}}q^{2n}}{2^{n+1}[n]!} \frac{H_{n+1}(\xi_\mu^\sigma)H_n(\xi_\nu^\tau) - H_{n+1}(\xi_\nu^\tau)H_n(\xi_\mu^\sigma)}{(\xi_\mu^\sigma - \xi_\nu^\tau)} \tag{2.8}$$

For the classical polynomials this identity has been used to prove completenes of the polynomials [6]. In the deformed case the question of completeness is still open.

In the mathematical literature the q-Hermite II polynomials $\tilde{h}_n$ have been studied [3]. They are related to our polynomials as follows:

$$H_n^{(q)}(\xi) = \frac{q^{-n^2} 2^{\frac{n}{2}}}{(1-q^{-2})^{\frac{n}{2}}} \tilde{h}_n(x';q^{-2}) \tag{2.9}$$

with the rescalation:

$$x' = \sqrt{2(q-q^{-1})}\,\xi$$

3 Eigenstates of the q-deformed harmonic oscillator

Here we are going to exploit the fact that the Hermite polynomials are part of the Eigenfunctions of the Hamiltonian of the q-deformed harmonic oscillator [1]. This oscillator is realized in the Hilbert space of the q-deformed Heisenberg algebra:

$$q^{\frac{1}{2}}XP - q^{-\frac{1}{2}}PX = iU \tag{3.1}$$

The momentum operator has the following eigenvectors and eigenstates:

$$P|l,\sigma\rangle = \sigma q^l |l,\sigma\rangle$$
$$l = -\infty \ldots \infty, \qquad \sigma = \pm 1 \tag{3.2}$$
$$\langle l',\sigma'|l,\sigma\rangle = \delta_{l'l}\delta_{\sigma'\sigma}$$

For the coordinates we find:

$$X|\nu,\tau\rangle = -\tau \frac{q^{\nu-\frac{1}{2}}}{q-q^{-1}}|\nu,\tau\rangle$$
$$\nu = -\infty \ldots \infty, \qquad \tau = \pm 1 \tag{3.3}$$
$$\langle \nu',\tau'|\nu,\tau\rangle = \delta_{\nu'\nu}\delta_{\tau'\tau}$$

The operator U acts on these states as follows:

$$U|l,\sigma\rangle = |l-1,\sigma\rangle \qquad \text{momentum}$$
$$U|\nu,\tau\rangle = |\nu+1,\tau\rangle \qquad \text{coordinates} \tag{3.4}$$

These two systems of eigenfunctions are related by the q-Fourier transformation:

$$|2l,\sigma\rangle = \frac{N_q}{2} \sum_{\substack{\nu=-\infty \\ \tau=+,-}}^{\infty} q^{\nu+l} \Big\{ \cos_q 2(\nu+l) - i\sigma\tau \sin_q 2(\nu+l) U \Big\} |2\nu,\tau\rangle$$
$$|2l+1,\sigma\rangle = U^{-1}|2l,\sigma\rangle \tag{3.5}$$

The q-trigonometric functions are:

$$\cos_q(2\nu) \equiv \cos(q^{2\nu};q^{-4})$$
$$\sin_q(2\nu) \equiv \sin(q^{2\nu};q^{-4}) \tag{3.6}$$

and

$$N_q \equiv \frac{(q^{-2};q^{-4})_\infty}{(q^{-4};q^{-4})_\infty} \tag{3.7}$$

This is in the notation defined in [7].The q-trigonometric functions satisfy the completeness and orthogonality relations:

$$\sum_{n=-\infty}^{+\infty} q^{-2n} \cos_q(-2(k+n)) \cos_q(-2(l+n)) = \frac{1}{N_q^2} q^{2l} \delta_{kl} \tag{3.8}$$
$$\sum_{n=-\infty}^{+\infty} q^{-2n} \sin_q(-2(k+n)) \sin_q(-2(l+n)) = \frac{1}{N_q^2} q^{2l} \delta_{kl}$$

The eigenfunctions of the harmonic oscillator have been defined in [1]. They are degenerate:

$$|n\rangle^r = \frac{1}{\sqrt{2^n[n]!}} H_n^{(q)}(X)|0\rangle^r \tag{3.9}$$

with

$$n = 0,1,\ldots\infty, \qquad r = 0,1$$

The polynomials $H_n^{(q)}(X)$ are functions of the coordinate operator X. The ground state, however, is easy to define in the momentum representation (3.2):

$$|0\rangle^r = \frac{1}{\sqrt{2}} \sum_{\substack{l=-\infty \\ \sigma=+,-}}^{\infty} (-1)^l \sigma^{l+r} q^{-\frac{1}{2}(l^2+l)} c_0 |l,\sigma\rangle \tag{3.10}$$

We can Fourier transform these ground states to the X basis using (3.5). With the definition:

$$c_l = q^{-\frac{1}{2}(l^2+l)} c_0$$

the Fourier coefficients are:

$$\langle 2\nu,\tau|0\rangle^0 = \frac{N_q}{\sqrt{2}} \sum_{l=-\infty}^{\infty} q^{\nu+l} \Big(c_{2l} \cos_q 2(\nu+l) + i\tau c_{2l+1} \sin_q 2(\nu+l) \Big)$$
$$\langle 2\nu+1,\tau|0\rangle^0 = 0 \tag{3.11}$$

and:

$$\langle 2\nu+1,\tau|0\rangle^1 = -\frac{N_q}{\sqrt{2}}\sum_{l=-\infty}^{\infty} q^{\nu+l}\Big(c_{2l+1}q\cos_q 2(\nu+l+1) + i\tau c_{2l}\sin_q 2(\nu+l)\Big)$$

$$\langle 2\nu,\tau|0\rangle^1 = 0 \tag{3.12}$$

4 Orthogonality relations

The eigenstates (3.9) are orthogonal.

$$\begin{aligned}\delta_{nm}\delta_{rr'} &= {}^{r'}\langle n|m\rangle^r \\ &= K_{nm}\,{}^{r'}\langle 0|H_n^{(q)}(X)H_m^{(q)}(X)|0\rangle^r \\ &= K_{nm}\sum_{\substack{\nu=-\infty\\ \tau=+,-}}^{\infty} H_n^{(q)}(\xi_{\nu,\tau})H_m^{(q)}(\xi_{\nu,\tau})\,{}^{r'}\langle 0|\nu,\tau\rangle\langle\nu,\tau|0\rangle^r\end{aligned} \tag{4.1}$$

with

$$\xi_{\nu,\tau} = -\tau\frac{q^{\nu-\frac{1}{2}}}{q-q^{-1}}$$

and

$$K_{nm} = \frac{1}{\sqrt{2^{n+m}[n]![m]!}}$$

We see that depending on r only the even or odd integers ν contribute to the sum and we obtain two orthogonality relations:

$$\sum_{\substack{\nu=-\infty\\ \tau=+,-}}^{\infty} H_n^{(q)}(\xi_{2\nu,\tau})H_m^{(q)}(\xi_{2\nu,\tau})|\langle 2\nu,\tau|0\rangle^0|^2 = 2^n[n]!\delta_{nm}$$

$$\sum_{\substack{\nu=-\infty\\ \tau=+,-}}^{\infty} H_n^{(q)}(\xi_{2\nu+1,\tau})H_m^{(q)}(\xi_{2\nu+1,\tau})|\langle 2\nu+1,\tau|0\rangle^1|^2 = 2^n[n]!\delta_{nm} \tag{4.2}$$

These are two orthogonality relations for the q-deformed Hermite polynomials with the two measures:

$$\mu^0(2\nu) = |\langle 2\nu,\tau|0\rangle^0|^2 \qquad \mu^0(2\nu+1) = 0$$
$$\mu^1(2\nu+1) = |\langle 2\nu+1,\tau|0\rangle^1|^2 \qquad \mu^1(2\nu) = 0 \tag{4.3}$$

These measures are independant of τ. For the q-Hermite II polynomials $\tilde{h}_n$ of the mathematical literature (cf. (2.9)) the following orthogonality relation is given [3]:

$$\tilde{N}_q\frac{(q;q)_n}{q^{n^2}}\delta_{nm} = \sum_{k=-\infty}^{\infty}\Big[\tilde{h}_n(q^k;q)\tilde{h}_m(q^k;q) + \tilde{h}_n(-q^k;q)\tilde{h}_m(-q^k;q)\Big]\omega(q^k)q^k \tag{4.4}$$

here $\tilde{N}_q$ is a normalisation constant independent of n and the summation is over all numbers. The measure is given by:

$$\omega(q^k) = \frac{1}{(-q^{2k};q^2)_\infty} \tag{4.5}$$

5 The moment functional

The groundstate expectation value of ξ^n can be computed. We proceed as follows: First we expand ξ^n in terms of the q-Hermite polynomials:

$$\xi^n = \sum_{k=0}^{n} b_k^{(n)} H_k^{(q)}(\xi) \tag{5.1}$$

The exited states of the harmonic oscillator are given in terms of the Hermite polynomials (3.9). They are orthogonal to the groundstate. We conclude:

$${}^r\langle 0|\xi^n|0\rangle^r = b_0^{(n)} \tag{5.2}$$

This is independent of r. With the help of the generating function of the q-Hermite polynomials (2.6) it is possible to calculate the coefficients $b_0^{(n)}$ explicitly. Inserting the definition of the q-exponentials into (2.6) we get:

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{q^{-2\binom{k}{2}}(\xi t)^k}{(q^{-2};q^{-2})_k} &= \sum_{n=0}^{\infty}\frac{q^{\frac{n}{2}}2^{-n}H_n^{(q)}(\xi)t^n}{(q^{-2};q^{-2})_n}\sum_{j=0}^{\infty}\frac{q^{-4\binom{j}{2}}(-q)^j t^{2j}}{(q^{-4};q^{-4})_j 2^j(1-q^2)^j} \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{q^{\frac{n-j}{2}}2^{-(n-j)}H_{n-j}^{(q)}(\xi)t^{n-j}q^{-4\binom{j}{2}}(-1)^j q^j t^{2j}}{(q^{-2};q^{-2})_{n-j}(q^{-4};q^{-4})_j 2^j(1-q^2)^j} \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{<\frac{k}{2}]}\frac{q^{\frac{k}{2}}2^{-(k-2j)}H_{k-2j}^{(q)}(\xi)q^{-4\binom{j}{2}}(-1)^j}{(q^{-2};q^{-2})_{k-2j}(q^{-4};q^{-4})_j 2^j(1-q^2)^j}t^k\end{aligned} \tag{5.3}$$

For the last step we changed the summation over n to the summation over k with $n=k-j$. On both sides are polynomials in t. Comparing the coefficients yields:

$$\frac{q^{-2\binom{k}{2}}\xi^k}{(q^{-2};q^{-2})_k} = \sum_{j=0}^{<\frac{k}{2}]}\frac{q^{\frac{k}{2}}2^{-k+j}q^{-4\binom{j}{2}}(-1)^j}{(q^{-2};q^{-2})_{k-2j}(q^{-4};q^{-4})_j(1-q^2)^j}H_{k-2j}^{(q)}(\xi)$$

and therefore

$$\xi^k = \sum_{j=0}^{<\frac{k}{2}]}\frac{q^{\frac{k}{2}}2^{-k+j}q^{-4\binom{j}{2}}q^{2\binom{k}{2}}(-1)^j(q^{-2};q^{-2})_k}{(q^{-2};q^{-2})_{k-2j}(q^{-4};q^{-4})_j(1-q^2)^j}H_{k-2j}^{(q)}(\xi) \tag{5.4}$$

This is the linear combination (5.1). Putting the different powers of q together we finally get:

$$\begin{aligned}\xi^k &= \sum_{j=0}^{<\frac{k}{2}]}\frac{q^{\frac{k}{2}}2^{-k+j}q^{-2j(j-1)}q^{k(k-1)}(-1)^j(q^{-2};q^{-2})_k}{(q^{-2};q^{-2})_{k-2j}(q^{-4};q^{-4})_j(-q^2)^j(1-q^{-2})^j}\times H_{k-2j}^{(q)}(\xi) \\ &= \sum_{j=0}^{<\frac{k}{2}]}\frac{q^{k^2-2j^2-\frac{k}{2}}2^{j-k}(q^{-2};q^{-2})_k}{(q^{-2};q^{-2})_{k-2j}(q^{-4};q^{-4})_j(1-q^{-2})^j}\times H_{k-2j}^{(q)}(\xi)\end{aligned} \tag{5.5}$$

We see that for odd powers ξ^{2n+1} the coefficients $b_0^{(2n+1)}$ vanish. For even powers we get:

$$b_0^{(2n)} = \frac{q^{\binom{2n}{2}}(q^{-2};q^{-2})_{2n}}{2^n(q^{-4};q^{-4})_n(1-q^{-2})^n} = \frac{q^{\binom{2n}{2}}}{2^n}[1][3]\dots[2n-1] \quad (5.6)$$

We know how to compute the groundstates in a basis where ξ is diagonal, the coefficients for the expansion have been calculated and are explicitly given by (3.11, 3.12). We obtain:

$$b_0^{(2n)} = \sum_{\substack{\nu=-\infty \\ \tau=+,-}}^{\infty} (\xi_{\nu,\tau})^{2n} |\langle \nu,\tau|0\rangle^r|^2 \quad (5.7)$$

More explicitely we find for $r = 0$:

$$b_0^{(2n)} = N_q^2 \sum_{\nu,j,k=-\infty}^{\infty} (\xi_{2\nu,\tau})^{2n} q^{2\nu+j+k} \cdot\big(c_{2j}c_{2k}\cos_q 2(\nu+j)\cos_q 2(\nu+k) + c_{2j+1}c_{2k+1}\sin_q 2(\nu+j)\sin_q 2(\nu+k)\big) \quad (5.8)$$

and for $r = 1$:

$$b_0^{(2n)} = N_q^2 \sum_{\nu,j,k=-\infty}^{\infty} (\xi_{2\nu+1,\tau})^{2n} q^{2\nu+j+k} \cdot\big(c_{2j}c_{2k}\sin_q 2(\nu+j)\sin_q 2(\nu+k) + c_{2j+1}c_{2k+1}q^2\cos_q 2(\nu+j+1)\cos_q 2(\nu+k+1)\big) \quad (5.9)$$

The expansion (5.7) can be interpreted as an integral with the measures:

$$d_q\mu^r(\xi) = |\langle \nu,\tau|0\rangle^r|^2 \quad (5.10)$$

These are two different measures, for $r = 0$ the measure is different from zero only for even values of ν, for $r = 1$ only for odd values of ν.

We have obtained the following moment functionals:

$$\mathcal{L}[\xi^{2n}] = \int \xi^{2n}\, d_q\mu^r(\xi) = \frac{q^{\binom{2n}{2}}}{2^n}[1][3]\dots[2n-1]$$
$$\mathcal{L}[\xi^{2n+1}] = \int \xi^{2n+1}\, d_q\mu^r(\xi) = 0 \quad (5.11)$$

Although we have two different measures the calculation shows that the moment functional is independent of the specific measure. All that enters into the moment functional is the normalisation of the measure.

That is exactly what is stated by Favards theorem [8]. It postulates the existence of a unique moment functional for any polynomial sequence that is given by a three-term recurrence relation without saying anything about the measure, not even about a possible uniqueness.

All classical orthogonal polynomials are orthogonal with respect to a unique measure, but for q-polynomials this is not the case (e.g. the q-Laguerre polynomials) [8]. It seems that by a q-quantum mechanical argumentation we have found another example.

5.1 The q-gamma function

In this section we want to give the result of the last section – the moment functional – in terms of a q-deformed gamma function: $\Gamma_q(x)$. This function is defined by $(0 < q < 1)$ [8]:

$$\Gamma_q(x) \equiv \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} \quad (5.12)$$

In [8] also the classical limit $q \to 1$ to the undeformed gamma function and some of its properties are given.

With the help of the identity:

$$\left[\frac{n}{2}\right]_{q^{-4}} = \frac{1-q^{-2n}}{1-q^{-4}} = \frac{1-q^{-2n}}{1-q^{-2}}\frac{1-q^{-2}}{1-q^{-4}} = \frac{[n]_{q^{-2}}}{[2]_{q^{-2}}} = \frac{[n]}{[2]} \quad (5.13)$$

and the functional equation for the q-gamma function we find the result:

$$\Gamma_{q^{-4}}\Big(\frac{2n+1}{2}\Big) = \left[\frac{2n-1}{2}\right]_{q^{-4}}\left[\frac{2n-3}{2}\right]_{q^{-4}}\dots\left[\frac{1}{2}\right]_{q^{-4}}\Gamma_{q^{-4}}\Big(\frac{1}{2}\Big) = \frac{[2n-1][2n-3]\dots[3][1]\Gamma_{q^{-4}}(\frac{1}{2})}{[2]^n} \quad (5.14)$$

Hence the moment functional for the q-Hermite polynomials can be expressed by the q-gamma function:

$$\int \xi^{2n}\, d_q\mu^r(\xi) = \frac{q^{\binom{2n}{2}}}{2^n}\frac{[2]^n}{\Gamma_{q^{-4}}(\frac{1}{2})}\Gamma_{q^{-4}}\Big(\frac{2n+1}{2}\Big)$$
$$\int \xi^{2n+1}\, d_q\mu^r(\xi) = 0$$

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